

PSEUDO-REAL RATIONAL MAPS AND THEIR AUTOMORPHISMS

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ABSTRACT. A rational map is called real if it is definable over the reals, equivalently, if it admits a reflection as an anticonformal automorphism. A rational map is called pseudo-real if it admits antiholomorphic automorphisms but it is not real. Real and pseudo-real maps are exactly those rational maps with the property that the field of moduli (the intersection of all fields of definition) is a subfield of the reals. Pseudo-real rational maps are the only known examples of rational maps which cannot be defined over their field of moduli. We observe that the group of holomorphic automorphisms of a pseudo-real rational map is either trivial or cyclic and we provide a characterization of pseudo-real rational maps in terms of the group of holomorphic automorphisms. The non-connectivity in moduli space of the locus of pseudo-real rational maps is also obtained.

1. INTRODUCTION

In this paper we are concerned with rational maps ϕ which are conjugated to their conjugate $\bar{\phi}$ (which is obtained from ϕ after conjugating its coefficients). There are two types of such rational maps: (i) the real ones (which are definable over the reals) and (ii) the pseudo-real ones (these are also called quasi-real). The pseudo-real rational maps are the only known examples of rational maps which cannot be defined over their field of moduli (the intersection of all the fields of definition) and also they must necessarily have odd order $d \geq 3$ and cannot be conjugated to a polynomial [9].

A rational map is real if and only if it admits a reflection as antiholomorphic automorphism and it is pseudo-real if it is non-real and it admits antiholomorphic automorphisms.

The group of holomorphic automorphisms of a rational map of degree $d \geq 2$ is a finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$; so it can be either trivial or isomorphic to a cyclic group or to a dihedral group or either of the alternating groups \mathcal{A}_4 and \mathcal{A}_5 or isomorphic to the symmetric group \mathfrak{S}_4 . In any of these possibilities there are rational maps with the corresponding group of holomorphic automorphisms [7]. For pseudo-real rational maps the situation is different. We observe that the group of holomorphic automorphisms of a pseudo-real rational map is either trivial or cyclic (see Theorem 1).

We also provide a characterization of pseudo-real rational maps in terms of its group of holomorphic automorphisms (see Theorems 2, 3). As a consequence of the characterization, we are able to note that the locus in moduli space of pseudo-real rational maps is non-connected (see Theorem 5).

Section 2 we recall some basic facts on rational maps. In Section 3 we provide the main results of this paper. A simple proof of Theorem 1 is provided. In Section 4 we recall some known results about holomorphic automorphisms of rational maps (see, for instance, [7]). In Section 5 we study those rational maps admitting an antiholomorphic automorphism and compute the corresponding

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real dimensions in the moduli space. In particular, we obtain that the locus in M_d consisting of classes of pseudo-real maps admitting an imaginary reflection as automorphism is connected and of real dimension $2d - 2$ and we provide a proof of Theorem 5. Section 6 provides the proves of Theorems 2 and 3.

2. PRELIMINARIES

2.1. Rational maps and their automorphisms. The space Rat_d , of complex rational maps of degree d , can be identified with the Zariski open set $\mathbb{P}_{\mathbb{C}}^{2d+1} \setminus \text{Res}_d$, where Res_d is the algebraic hypersurface defined by the resultant of two polynomials of degree at most d . The identification is given by

$$\Xi : \text{Rat}_d \rightarrow \mathbb{P}_{\mathbb{C}}^{2d+1} \setminus \text{Res}_d$$

$$\phi(z) = \frac{\sum_{k=0}^d a_k z^k}{\sum_{k=0}^d b_k z^k} \mapsto \Xi(\phi) = [a_0 : \cdots : a_d : b_0 : \cdots : b_d]$$

The group $\text{PSL}_2(\mathbb{C})$, acting on the Riemann sphere $\widehat{\mathbb{C}}$ as group of Möbius transformations (this is the full group of holomorphic automorphisms of $\widehat{\mathbb{C}}$), acts on Rat_d by conjugation: $\phi, \psi \in \text{Rat}_d$ are said to be equivalent if there is some $T \in \text{PSL}_2(\mathbb{C})$ so that $\psi = T \circ \phi \circ T^{-1}$; we use the notation $\phi \sim \psi$. The conjugation action of $\text{PSL}_2(\mathbb{C})$ on Rat_d turns-out to be a linear action on the identified model $\mathbb{P}_{\mathbb{C}}^{2d+1} \setminus \text{Res}_d$. The quotient space $M_d = \text{Rat}_d / \text{PSL}_2(\mathbb{C})$ is called the moduli space of rational maps of degree d . Silverman [10] proved that M_d carries the structure of an affine geometric quotient and Milnor [8] proved it carries the structure of a complex orbifold of dimension $2(d - 1)$ (Milnor also proved that $M_2 \cong \mathbb{C}^2$). It seems that, for $d \geq 3$, no explicit model is known for M_d . The $\text{PSL}_2(\mathbb{C})$ -stabilizer of $\phi \in \text{Rat}_d$ (that is, the collection of Möbius transformations commuting with ϕ) is called the group of holomorphic automorphisms of ϕ and it is denoted as $\text{Aut}(\phi)$. This group is known to be a finite group if $d \geq 2$ (this is consequence of the fact that a Möbius transformation is uniquely determined by its action at three different points). These finite subgroups of $\text{PSL}_2(\mathbb{C})$ are either the trivial group or isomorphic to a cyclic group, dihedral group, the alternating groups $\mathcal{A}_4, \mathcal{A}_5$ or the symmetric group \mathfrak{S}_4 [1].

The antiholomorphic automorphisms of $\widehat{\mathbb{C}}$ are given by the extended Möbius transformations, that is, transformations of the form $Q = T \circ J$, where $T \in \text{PSL}_2(\mathbb{C})$ and $J(z) = \bar{z}$. An extended Möbius transformation of order two is called a reflection (respectively, an imaginary reflection) if it has fixed points (respectively, has no fixed points) on the Riemann sphere. Each reflection (respectively, imaginary reflection) is conjugated by a suitable Möbius transformation to $J(z) = \bar{z}$ (respectively, $\tau(z) = -1/\bar{z}$). The full group of holomorphic and antiholomorphic automorphisms of $\widehat{\mathbb{C}}$, denoted by $\widehat{\text{PSL}}_2(\mathbb{C})$, is generated by $\text{PSL}_2(\mathbb{C})$ and the reflection J . It again acts by conjugation on Rat_d . The $\widehat{\text{PSL}}_2(\mathbb{C})$ -stabilizer of $\phi \in \text{Rat}_d$ is called the group of holomorphic and antiholomorphic automorphisms of ϕ and it will be denoted as $\widehat{\text{Aut}}(\phi)$; the elements in $\widehat{\text{Aut}}(\phi) \setminus \text{Aut}(\phi)$ are the antiholomorphic automorphisms of ϕ .

2.2. Real and pseudo-real rational maps. Let us observe that the action of $J(z) = \bar{z}$ on Rat_d corresponds to the canonical conjugation in the model $\Xi(\text{Rat}_d) = \mathbb{P}_{\mathbb{C}}^{2d+1} \setminus \text{Res}_d$

$$\begin{array}{ccc} \phi \in \text{Rat}_d & \longrightarrow & [a_0 : \cdots : b_d] \in \Xi(\text{Rat}_d) \subset \mathbb{P}_{\mathbb{C}}^{2d+1} \\ \downarrow & & \downarrow \\ J \circ \phi \circ J \in \text{Rat}_d & \longrightarrow & [\overline{a_0} : \cdots : \overline{b_d}] \in \Xi(\text{Rat}_d) \subset \mathbb{P}_{\mathbb{C}}^{2d+1} \end{array}$$

As J normalizes $\mathrm{PSL}_2(\mathbb{C})$, the above action induces a real structure \widehat{J} on M_d , that is, an antiholomorphic automorphism of order two. The real points of such a structure, that is, the fixed points of \widehat{J} are given by those classes $[\phi]$ so that $\phi \sim \bar{\phi} = J \circ \phi \circ J$ (note that $\bar{\phi}$ is obtained from ϕ by applying conjugation to all its coefficients). We may see that the real points are exactly those $[\phi]$ so that ϕ admits antiholomorphic automorphisms.

If ϕ admits a reflection as anticonformal automorphism, then ϕ is real [3], that is, there is some $\psi \in \mathrm{Rat}_d$ with coefficients in \mathbb{R} and $\psi \sim \phi$. The rational maps which are not real but with $[\phi]$ being a real point are called pseudo-real.

A subfield K of \mathbb{C} is called a field of definition of $\phi \in \mathrm{Rat}_d$ if there is some $\psi \in \mathrm{Rat}_d$ whose coefficients belong to K with $\psi \sim \phi$; we say that ϕ is definable over K and that ψ is defined over K . The intersection of all fields of definitions of ϕ is called the field of moduli of it. For instance, the quadratic polynomial $\phi_c(z) = z^2 + c$, where $c \in \mathbb{C}$, has field of moduli $\mathbb{Q}(c)$; which in this case is a field of definition (this comes from the fact that $\phi_c \sim \phi_d$ if and only if $c = d$). Real and pseudo-real rational maps are the only ones whose field of moduli is a subfield of \mathbb{R} .

The only known examples of rational maps which cannot be defined over their field of moduli are pseudo-real ones. This may be a reason to think that pseudo-real rational maps can be of interest. Explicit examples of pseudo-real rational maps, provided by Silverman in [9], are

$$\phi_d(z) = i \left(\frac{z-1}{z+1} \right)^d, \quad d \geq 3 \text{ odd}.$$

We may see that the imaginary reflection $Q(z) = -1/\bar{z}$ is an antiholomorphic automorphism of ϕ_d . In order to see that ϕ_d admits no reflection R as automorphism, one may proceed as follows. Since ± 1 are the only critical points of ϕ_d , it must happen that either (i) $R(1) = 1$ and $R(-1) = -1$ or (ii) $R(1) = -1$ and $R(-1) = 1$ (since R must send critical points onto critical points). In case (i) the equality $R \circ \phi_d \circ R = \phi_d$ then ensures that each point in the future orbit of 1 must be a fix point of R . In particular, these orbit must be contained in the circle of fixed points of R . The first points in that orbit are 1, 0, $-i$ and $i(i+1)^d/(1-i)^d$; which clearly do not belong to a common circle (the cross-ratio is not real). In case (ii) the equality $R \circ \phi_d \circ R = \phi_d$ then ensures that $R(0) = \infty$ and $R(i) = -1$. This implies that $R(z) = -1/\bar{z}$, which is not a reflection.

In [9] Silverman proved that the field of moduli is a field of definition if either (i) the degree of ϕ is even or (ii) ϕ is equivalent to a polynomial. In particular, pseudo-real rational maps must have odd degree $d \geq 3$.

Let us note that Silverman's pseudo-real rational maps ϕ_d have field of moduli \mathbb{Q} (which we know it is not a field of definition), but they are defined over an extension of degree two of it. In [3] it was proved that this is the general situation, that is, every rational map, which cannot be defined over its field of moduli, can be defined over an extension of degree two of it.

We conjecture the following.

Conjecture 1. *Every real rational map is definable over its field of moduli.*

A related conjecture is the following.

Conjecture 2. *A real rational map admitting $\overline{\mathbb{Q}}$ as a field of definition can be defined over $\mathbb{R} \cap \overline{\mathbb{Q}}$.*

The above conjecture is true for rational maps of degree at most one. Assume that ϕ has degree $d \geq 2$, is defined over $\overline{\mathbb{Q}}$ and it is real. We know that ϕ must admit a reflection R as antiholomorphic automorphism. As the group of automorphisms of ϕ is finite, R is defined over $\overline{\mathbb{Q}}$. This permits to construct a birational isomorphism $\Theta : \widehat{\mathbb{C}} \rightarrow X$, defined over $\overline{\mathbb{Q}}$, so that X is a (smooth) genus zero curve defined over $\mathbb{R} \cap \overline{\mathbb{Q}}$ with the property that $\Theta \circ \phi \circ \Theta^{-1}$ is also defined over $\mathbb{R} \cap \overline{\mathbb{Q}}$. So, if X has some $(\mathbb{R} \cap \overline{\mathbb{Q}})$ -rational point, then (as a consequence of the Riemann-Roch theorem) there will be an isomorphism $\Psi : X \rightarrow \widehat{\mathbb{C}}$, defined over $\mathbb{R} \cap \overline{\mathbb{Q}}$. Then $L = \Psi \circ \Theta$ will be a Möbius transformation (defined over the previous field) so that $L \circ \phi \circ L^{-1}$ is defined over $\mathbb{R} \cap \overline{\mathbb{Q}}$. This will provide a proof of the above conjecture. Unfortunately, the existence of the $(\mathbb{R} \cap \overline{\mathbb{Q}})$ -rational point is not clear.

3. MAIN RESULTS

3.1. Characterization of pseudo-real maps. We provide a characterization of pseudo-real rational maps in terms of the group of holomorphic automorphisms. Our first observation ensures that a pseudo-real rational map cannot have group of automorphisms different from trivial or cyclic.

Theorem 1. *If $\phi \in \text{Rat}_d$ is pseudo-real, then $\text{Aut}(\phi)$ is either trivial or cyclic.*

Proof. Let $\phi \in \text{Rat}_d$ be a pseudo-real rational map, so $d \geq 3$ is odd and $\text{Aut}(\phi)$ is a finite group. Let us assume $G = \text{Aut}(\phi)$ is either a dihedral group or isomorphic to \mathcal{A}_4 , \mathcal{A}_5 or \mathfrak{S}_4 . Then there is a branched regular cover $\pi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ whose branch values are ∞ , 0 and 1 . Let $Q \in \widehat{\text{Aut}}(\phi)$ be an antiholomorphic automorphism of ϕ . Since Q normalizes $\text{Aut}(\phi)$, there is an extended Möbius transformation \widehat{Q} of order two so that $\pi \circ Q = \widehat{Q} \circ \pi$. In particular, as \widehat{Q} must keep invariant the branch locus $\{\infty, 0, 1\}$ and it has order two, \widehat{Q} must have a fixed point; so it is a reflection. But this implies that some lifting of \widehat{Q} must have a continuum of fixed points; so it must be a reflection, a contradiction. \square

Next, a characterization of a pseudo-real rational map, in terms of the different possibilities for its group of holomorphic automorphisms, is provided.

The first case, when the rational map has no non-trivial holomorphic automorphisms, the pseudo-real property is easy to state (this has been also observed by Silverman [12]).

Theorem 2. *Let $\phi \in \text{Rat}_d$, $d \geq 3$ odd, so that $\text{Aut}(\phi)$ is trivial. Then ϕ is pseudo-real if and only if there is an imaginary reflection being an automorphism of ϕ .*

As every imaginary reflection is conjugated by a suitable Möbius transformation to $\tau(z) = -1/\overline{z}$, the previous result can be written as follows.

Corollary 1. *Let $\phi \in \text{Rat}_d$, $d \geq 3$ odd, so that $\text{Aut}(\phi)$ is trivial. Then ϕ is pseudo-real if and only if there is a Möbius transformation $T \in \text{PSL}_2(\mathbb{C})$ so that $\eta = T \circ \phi \circ T^{-1}$ satisfies that $\eta(-1/\overline{\eta(z)}) = -1/\overline{\eta(z)}$.*

In the case that the group of automorphisms of ϕ is the cyclic group \mathbb{Z}_n , where $n \geq 2$, then (see Section 4) we may assume, up to conjugation by a suitable Möbius transformation, that $\phi(z) = z\psi(z^n)$, for a suitable rational map ψ , and the cyclic group is generated by the rotation $T(z) = \omega_n z$, where $\omega_n = e^{2\pi i/n}$. We obtain the following.

Theorem 3. *Let $\phi \in \text{Rat}_d$, $d \geq 3$ odd, with $\text{Aut}(\phi) \cong \mathbb{Z}_n$ for $n \geq 2$. Then the following are equivalent.*

- (1) ϕ is pseudo-real.
- (2) ϕ is conjugated to a rational map $\tilde{\phi}(z) = z\psi(z^n)$ so that
 - (a) $\psi(z) \neq \bar{\psi}(e^{i\theta}z)$, for every $\theta \in \mathbb{R}$ and
 - (b) there is some $\alpha = e^{i\varphi}$ with $\varphi \neq 2s\pi/n$, for every $s \in \{0, 1, \dots, n-1\}$, so that $\psi(z)\bar{\psi}(\alpha^n/z) = 1$.

3.2. Explicit examples of pseudo-real rational maps with non-trivial holomorphic automorphisms. Let $n \geq 6$ be an integer and let $\psi(z)$ a rational map satisfying the following three properties.

- (1) $\psi(z)$ cannot be written as a rational map of the form $\rho(z^m)$, for some $m \geq 2$.
- (2) For every $s \in \mathbb{C} - \{0\}$, the rational maps $\psi(z)$ and $\frac{1}{\psi(s/z)}$ are different.
- (3) $\bar{\psi}(z) = \frac{1}{\psi(-1/\bar{z})}$.

Set $\phi(z) = z\psi(z^n)$.

As $n \geq 6$, $\text{Aut}(\phi)$ is either cyclic or dihedral. Moreover $\langle T(z) = \omega_n z \rangle < \text{Aut}(\phi)$.

Condition (2) ensures that $\text{Aut}(\phi)$ cannot be dihedral.

Condition (1) ensures that $\text{Aut}(\phi) = \langle T \rangle$.

Condition (3) ensures that $\tau_n(z) = \omega_{2n}/\bar{z}$ is antiholomorphic automorphism of ϕ .

All antiholomorphic automorphisms of ϕ then will be of the form

$$T^l \circ \tau_n(z) = \omega_n^l \omega_{2n} / \bar{z}$$

As $\omega_n^l \omega_{2n} \neq 1$, none of them is a reflection. As a consequence, ϕ is a pseudo-real map with $\text{Aut}(\phi) \cong \mathbb{Z}_n$.

Now, let us construct some explicit examples. Let us write

$$\psi(z) = \frac{\sum_{k=0}^r a_k z^k}{\sum_{k=0}^r b_k z^k}$$

where we assume $a_r \neq 0$ (so, in this case, ϕ will have degree $d = 1 + nr$).

Condition (3) is equivalent to have r even (so d will be odd as supposed to be) and the existence of some $\theta \in \mathbb{R}$ so that, for every $k = 0, 1, \dots, r$, it holds

$$b_k = (-1)^k e^{i\theta} \overline{a_{r-k}}.$$

Condition (2) will be satisfied if we also assume (this is by evaluation at $z = 0$)

$$a_0 a_r \neq e^{2i\theta} \overline{a_0 a_r}$$

Condition (1) will be satisfied if we assume

$$a_1 \neq 0$$

Summarizing all the above.

Theorem 4. *Let $n \geq 6$, $r \geq 2$ be even, $\theta \in \mathbb{R}$ and let $a_0, \dots, a_r \in \mathbb{C}$ be so that*

$$\begin{aligned} a_1 a_r &\neq 0, \\ a_0 a_r &\neq e^{2i\theta} \overline{a_0 a_r}. \end{aligned}$$

Set

$$\psi(z) = \frac{\sum_{k=0}^r a_k z^k}{\sum_{k=0}^r (-1)^k e^{i\theta} \overline{a_{r-k}} z^k}.$$

Then $\phi(z) = z\psi(z^n)$ is a pseudo-real rational map with $\text{Aut}(\phi) = \langle T(z) = \omega_n z \rangle \cong \mathbb{Z}_n$.

Example 1. *If, in Theorem 4, we set $n = 6$, $r = 2$, $\theta = 0$, $a_0 = 1$, $a_1 = 1$, $a_2 = i$, then*

$$\psi(z) = \frac{1 + z + iz^2}{-i - z + z^2},$$

and

$$\phi(z) = z \frac{1 + z^6 + iz^{12}}{-i - z^6 + z^{12}},$$

will be a pseudo-real rational map of degree 13 with $\text{Aut}(\phi) = \langle T(z) = e^{\pi i/3} z \rangle$.

3.3. On the connectivity of pseudo-real locus in moduli space. Let us denote by $\mathcal{B}_d^{\mathbb{R}}$ the sublocus in M_d consisting of the classes of equivalence of pseudo-real rational maps. By Silverman's results, for d even $\mathcal{B}_d^{\mathbb{R}} = \emptyset$. Silverman's examples assert that, for every $d \geq 3$ odd $\mathcal{B}_d^{\mathbb{R}} \neq \emptyset$. The following result states that, for $d \geq 3$ odd the previous locus is disconnected.

Theorem 5. *If $d \geq 3$ is odd, then $\mathcal{B}_d^{\mathbb{R}}$ is not connected.*

In [4] it was observed that the branch locus \mathcal{B}_d (the equivalence classes of rational maps with non-trivial holomorphic automorphisms) is connected.

3.4. Quotients of pseudo-real rational maps with non-trivial holomorphic automorphisms. Let us consider a pseudo-real rational map $\phi \in \text{Rat}_d$, $d \geq 3$ odd, whose group of holomorphic automorphisms is \mathbb{Z}_n , where $n \geq 2$.

Theorem 3 ensures that, up to conjugation by a suitable Möbius transformation, we may assume that $\phi(z) = z\psi(z^n)$ with the following properties:

- (i) $\psi(z) \neq \overline{\psi}(e^{i\theta} z)$, for every $\theta \in \mathbb{R}$, and
- (ii) there is some complex number $\alpha = e^{i\varphi}$ with $\varphi \neq 2s\pi/n$, for every $s \in \{0, 1, \dots, n-1\}$, so that $\psi(z)\overline{\psi}(\alpha^n/z) = 1$.

In this case, $\widehat{\text{Aut}}(\phi) = \langle \tau_n(z) = \omega_{2n}/\bar{z} \rangle \cong \mathbb{Z}_{2n}$.

Let us consider the branch regular cover $w = \pi(z) = z^n$ and the rational map $\widehat{\phi}(w) = w\psi(w)^n$.

It can be seen that $\pi \circ \phi = \widehat{\phi} \circ \pi$ and that $\widehat{\phi}$ admits the imaginary reflection $\tau(w) = -1/\bar{w}$ as antiholomorphic automorphism. Also, as ϕ sends $\{0, \infty\}$ into itself, then $\widehat{\phi}$ does the same.

Theorem 6. *The quotient rational map $\widehat{\phi}$ is pseudo-real with no non-trivial holomorphic automorphisms.*

Proof. The condition that the only holomorphic automorphisms of ϕ are the powers of T asserts that, for every integer $m \geq 2$, there is no rational map $\xi(z)$ satisfying that $\psi(z) = \xi(z^m)$. If $\widehat{\phi}$ has no non-trivial holomorphic automorphisms, then it is pseudo-real.

Let us assume it has non-trivial holomorphic automorphisms and let $V \in \text{PSL}_2(\mathbb{C})$ be a maximal order non-trivial holomorphic automorphism of $\widehat{\phi}$. Then V must keep invariant the set $\{0, \infty\}$; so either (i) $V(w) = \beta/w$ or (ii) $V(w) = \omega_s w$, where $\omega_s = e^{2\pi i/s}$ for some $s \geq 2$.

Assume that we are in case (i). If $|\beta| \neq 1$, then $W = (\tau \circ V)^2$ is holomorphic automorphism of $\widehat{\phi}$ of infinite order, a contradiction. So, $|\beta| = 1$ and $\widehat{\phi}$ admits a reflection $\eta(w) = e^{i\theta}\bar{w}$ as antiholomorphic automorphism. The condition $\eta \circ \widehat{\phi} \circ \eta = \widehat{\phi}$ is equivalent to have $\bar{\psi}(e^{-n\theta i}w) = \psi(w)$, a contradiction by Theorem 3.

Now, assume we are in case (ii). In this case, $\widehat{\phi}(w) = w\rho(w^s)$, for a suitable rational map ρ ; so $\psi(w)^n = \rho(w^s)$ (recall that $n \geq 2$ is even and $s \geq 2$). Then $\psi(w) = \xi(w^s)$, a contradiction. \square

4. RATIONAL MAPS WITH HOLOMORPHIC AUTOMORPHISMS

In this section we recall the well known characterization of rational maps with non-trivial holomorphic automorphisms [7]. As a matter of completeness, in the cyclic case, we provide the arguments (see Theorem 7).

Let $\phi \in \text{Rat}_d$, where $d \geq 2$, admitting non-trivial holomorphic automorphisms. As the group of holomorphic automorphisms of ϕ is a finite group of $\text{PSL}_2(\mathbb{C})$, we may assume that, up to conjugation, $\text{Aut}(\phi)$ is one of the following groups [1]:

- (1) $\langle T_n \rangle \cong \mathbb{Z}_n$;
- (2) $\langle T_n, A : T_n^n = A^2 = (T_n \circ A)^2 = I \rangle \cong D_n$;
- (3) $\langle T_3, B : T_3^3 = B^2 = (T_3 \circ A)^3 = I \rangle \cong \mathcal{A}_4$;
- (4) $\langle T_4, C : T_4^4 = C^2 = (T_4 \circ C)^3 = I \rangle \cong \mathfrak{S}_4$;
- (5) $\langle T_5, D : T_5^5 = D^2 = (T_5 \circ D)^3 = I \rangle \cong \mathcal{A}_5$;

where

$$\begin{aligned} T_n(z) &= \omega_n z, & \omega_n &= e^{2\pi i/n}, \\ A(z) &= 1/\bar{z}, \\ B(z) &= \frac{(\sqrt{3}-1)(z + (\sqrt{3}-1))}{2z - (\sqrt{3}-1)}, \\ C(z) &= \frac{(\sqrt{2}+1)(-z + (\sqrt{2}+1))}{z + (\sqrt{2}+1)}, \end{aligned}$$

$$D(z) = \frac{\left(1 + \sqrt{2 - \omega_5 - \omega_5^4}\right)\left(-z + \left(1 + \sqrt{2 - \omega_5 - \omega_5^4}\right)\right)}{(1 - \omega_5 - \omega_5^4)z + \left(1 + \sqrt{2 - \omega_5 - \omega_5^4}\right)}.$$

4.1. The cyclic holomorphic situation. Let us first observe that every rational map of the form $\phi(z) = z\psi(z^n)$ admits T as holomorphic automorphism. Next result states that every rational map admitting a holomorphic automorphism of order $n \geq 2$ is conjugated to one of the previous ones.

Theorem 7 ([7]). *Let $d, n \geq 2$ be integers. The group \mathbb{Z}_n acts as a group of holomorphic automorphisms of some rational map of degree d if and only if d is congruent to either $-1, 0, 1$ modulo n . Moreover, for such values, every rational map of degree d admitting \mathbb{Z}_n as a group of holomorphic automorphisms is equivalent to one of the form $\phi(z) = z\psi(z^n)$, where*

$$\psi(z) = \frac{\sum_{k=0}^r a_k z^k}{\sum_{k=0}^r b_k z^k} \in \text{Rat}_r,$$

satisfies that

- (a) $a_r b_0 \neq 0$, if $d = nr + 1$;
- (b) $a_r \neq 0$ and $b_0 = 0$, if $d = nr$;
- (c) $a_r = b_0 = 0$ and $b_r \neq 0$, if $d = nr - 1$.

Proof. Let ϕ be a rational map admitting a holomorphic automorphism of order n . By conjugating it by a suitable Möbius transformation, we may assume that such automorphism is the rotation $T(z) = \omega_n z$.

(1) Let us write $\phi(z) = z\rho(z)$. The equality $T \circ \phi \circ T^{-1} = \phi$ is equivalent to $\rho(\omega_n z) = \rho(z)$. Let

$$\rho(z) = \frac{U(z)}{V(z)} = \frac{\sum_{k=0}^l \alpha_k z^k}{\sum_{k=0}^l \beta_k z^k},$$

where either $\alpha_l \neq 0$ or $\beta_l \neq 0$ and $(U, V) = 1$.

The equality $\rho(\omega_n z) = \rho(z)$ is equivalent to the existence of some $\lambda \neq 0$ so that

$$\omega_n^k \alpha_k = \lambda \alpha_k, \quad \omega_n^k \beta_k = \lambda \beta_k.$$

By taking $k = l$, we obtain that $\lambda = \omega_n^l$. So the above is equivalent to have, for $k < l$,

$$\omega_n^{l-k} \alpha_k = \alpha_k, \quad \omega_n^{l-k} \beta_k = \beta_k.$$

So, if $\alpha_k \neq 0$ or $\beta_k \neq 0$, then $l - k \equiv 0 \pmod{n}$. As $(U, V) = 1$, either $\alpha_0 \neq 0$ or $\beta_0 \neq 0$; so $l \equiv 0 \pmod{n}$. In this way, if $\alpha_k \neq 0$ or $\beta_k \neq 0$, then $k \equiv 0 \pmod{n}$. In this way, $\rho(z) = \psi(z^n)$ for a suitable rational map $\psi(z)$.

(2) It follows from (1) that $\phi(z) = z\psi(z^n)$, for $\psi \in \text{Rat}_r$ and suitable r . We next provide relations between d and r . Let us write

$$\psi(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^r a_k z^k}{\sum_{k=0}^r b_k z^k},$$

where $(P, Q) = 1$ and either $a_r \neq 0$ or $b_r \neq 0$. In this way,

$$\phi(z) = \frac{zP(z^n)}{Q(z^n)} = \frac{z \sum_{k=0}^r a_k z^{kn}}{\sum_{k=0}^r b_k z^{kn}}.$$

Let us first assume that $Q(0) \neq 0$, equivalently, $\psi(0) \neq \infty$. Then $\phi(0) = 0$ and the polynomials $zP(z^n)$ and $Q(z^n)$ are relatively prime. If $\deg(P) \geq \deg(Q)$, then $r = \deg(P)$, $\psi(\infty) \neq 0$, $\phi(\infty) = \infty$ and $\deg(\phi) = 1 + nr$. If $\deg(P) < \deg(Q)$, then $r = \deg(Q)$, $\psi(\infty) = 0$, $\phi(\infty) = 0$ and $\deg(\phi) = nr$.

Let us now assume that $Q(0) = 0$, equivalently, $\psi(0) = \infty$. Let us write $Q(u) = u^l \widehat{Q}(u)$, where $l \geq 1$ and $\widehat{Q}(0) \neq 0$; so $\deg(Q) = l + \deg(\widehat{Q})$. In this case,

$$\phi(z) = \frac{P(z^n)}{z^{ln-1} \widehat{Q}(z^n)}$$

and the polynomials $P(z^n)$ (of degree $n\deg(P)$) and $z^{ln-1} \widehat{Q}(z^n)$ (of degree $n\deg(Q) - 1$) are relatively prime. If $\deg(P) \geq \deg(Q)$, then $r = \deg(P)$, $\psi(\infty) \neq 0$, $\phi(\infty) = \infty$ and $\deg(\phi) = nr$. If $\deg(P) < \deg(Q)$, then $r = \deg(Q)$, $\psi(\infty) = 0$, $\phi(\infty) = 0$ and $\deg(\phi) = nr - 1$.

Summarizing all the above, we have the following situations:

- (i) If $\phi(0) = 0$ and $\phi(\infty) = \infty$, then $\psi(0) \neq \infty$ and $\psi(\infty) \neq 0$; in particular, $d = nr + 1$. This case corresponds to have $a_r b_0 \neq 0$.
- (ii) If $\phi(0) = \infty = \phi(\infty)$, then $\psi(0) = \infty$ and $\psi(\infty) \neq 0$; in which case $d = nr$. This case corresponds to have $a_r \neq 0$ and $b_0 = 0$.
- (iii) If $\phi(0) = 0 = \phi(\infty)$, then $\psi(0) \neq \infty$ and $\psi(\infty) = 0$; in particular, $d = nr$. This case corresponds to have $a_r = 0$ and $b_0 \neq 0$. But if we conjugate ϕ by $A(z) = 1/z$ (which normalizes $\langle T(z) = \omega_n z \rangle$) in order to be in case (ii) above.
- (iv) If $\phi(0) = \infty$ and $\phi(\infty) = 0$, then $\psi(0) = \infty$ and $\psi(\infty) = 0$; in particular, $d = nr - 1$. This case corresponds to have $a_r = b_0 = 0$ (so $b_r \neq 0$ as ψ has degree r).

□

Theorem 7, for $d = 2$, asserts that the only possibilities are $(n, r) \in \{(2, 1), (3, 1)\}$; in other words, a rational map of degree two has holomorphic automorphisms of order either two or three.

Remark 1. Since $n \geq 2$, then (a) and (b) (respectively, (b) and (c)) are complementary. Also, (a) and (c) will be complementary for $n > 2$.

If we denote by $M_d(\mathbb{Z}_n)$ the locus of M_d consisting of the classes of rational maps admitting an holomorphic automorphism of order $n \geq 2$, then Theorem 7 provides the following dimension counting.

Corollary 2 ([7]). *Let $d, n \geq 2$ be integers. Then*

$$\dim_{\mathbb{C}}(M_d(\mathbb{Z}_n)) = \begin{cases} 2(d-1)/n, & d \equiv 1 \pmod{n} \\ (2d-n)/n, & d \equiv 0 \pmod{n} \\ 2(d+1-n)/n, & d \equiv -1 \pmod{n} \end{cases}$$

Proof. By Theorem 7, every rational map in Rat_d admitting a holomorphic automorphism of order $n \geq 2$ is conjugated to one of the form $\phi(z) = z\psi(z^n) \in \text{Rat}_d$; so $T \in \text{Aut}(\phi)$. The normalizer in $\text{PSL}_2(\mathbb{C})$ of $\langle T \rangle$ is the 1-complex dimensional group $N_n = \langle A_\lambda(z) = \lambda z, B(z) = 1/z : \lambda \in \mathbb{C} - \{0\} \rangle$. If $U \in N_n$, then $U \circ \phi \circ U^{-1}$ will also have T as a holomorphic automorphism. In fact,

$$A_\lambda \circ \phi \circ A_\lambda^{-1}(z) = z\psi(z^n/\lambda^n),$$

$$B \circ \phi \circ B(z) = z/\psi(1/z^n).$$

In this way, there is an action of N_n over Rat_r so that the orbit of $\psi(u)$ is given by the rational maps $\psi(u/t)$, where $t \in \mathbb{C} - \{0\}$, and $1/\psi(1/u)$. \square

Remark 2. For $n = 1$, the above states that $\dim_{\mathbb{C}}(\mathbf{M}_d(\mathbb{Z}_n)) = d - 1$ (the maximal possible dimension).

4.2. The non-cyclic holomorphic situation. In order for $\phi(z) = z\psi(z^n)$ to have more automorphisms, we need ψ to satisfy some extra conditions. In case (2), the equality $A \circ \phi \circ A = \phi$ asserts that $\psi(1/u)\psi(u) = 1$. Similarly, in case (3) we must have $B(z)\psi(B(z)^3) = B(z\psi(z^3))$, in case (4) we must have $C(z)\psi(C(z)^4) = C(z\psi(z^4))$ and in case (5) we must have $D(z)\psi(D(z)^5) = D(z\psi(z^5))$.

As we are interested in pseudo-real rational maps, we will not go into details in this case.

Remark 3. Inside \mathbf{M}_d there are two important loci; the singular locus \mathcal{S}_d and the branch locus \mathcal{B}_d (the equivalence classes of rational maps with non-trivial holomorphic automorphisms). If $d \geq 3$, then $\mathcal{S}_d = \mathcal{B}_d$ [7], $\mathcal{S}_2 = \emptyset$ [8] and \mathcal{B}_2 is a cubic curve. In [4] we prove that \mathcal{B}_d is connected.

5. RATIONAL MAPS WITH ANTIHOLOMORPHIC AUTOMORPHISMS

Let us now consider the collection of rational maps $\phi \in \text{Rat}_d$ admitting an antiholomorphic automorphisms τ_n of order $2n$, where $n \geq 1$. Again, up to conjugation by a suitable Möbius transformation, we may assume that

$$\tau_n(z) = \frac{\omega_{2n}}{\bar{z}}, \quad \omega_{2n} = e^{\pi i/n}.$$

5.1. case $n = 1$. In this case, we have two possibilities, either $\tau_1(z) = 1/\bar{z}$ (a reflection) or $\tau_1(z) = -1/\bar{z}$ (imaginary reflection).

5.1.1. The reflection case. We may conjugate again by a suitable Möbius transformation in order to assume that $\tau_1(z) = J(z) = \bar{z}$. With this normalization, we may see that ϕ is defined over \mathbb{R} [3]. In particular, the locus $\mathbf{M}_d^{\mathbb{R}}$ consisting of the classes of those rational maps admitting a reflection as an automorphism is connected of real dimension $2d - 2$.

5.1.2. The imaginary reflection situation. In this case the antiholomorphic automorphism is the imaginary reflection $\tau_1(z) = \tau(z) = -1/\bar{z}$.

Proposition 1. If we write

$$\phi(z) = \frac{\sum_{k=0}^d a_k z^k}{\sum_{k=0}^d b_k z^k} \in \text{Rat}_d,$$

then ϕ admits $\tau(z) = -1/\bar{z}$ as automorphism if and only if d is odd and there exists $\theta \in \mathbb{R}$ so that

$$b_k = (-1)^k e^{i\theta} \overline{a_{d-k}}$$

Proof. The equality $\tau \circ \phi = \phi \circ \tau$ is equivalent to

$$\frac{\sum_{k=0}^d -\overline{b_k} z^k}{\sum_{k=0}^d \overline{a_k} z^k} = \frac{\sum_{k=0}^d (-1)^{d-k} a_{d-k} z^k}{\sum_{k=0}^d (-1)^{d-k} b_{d-k} z^k}$$

which is also equivalent to the existence of some $\lambda \neq 0$ so that

$$\begin{aligned} \overline{b_k} &= \lambda (-1)^{d+1-k} a_{d-k} \\ \overline{a_k} &= \lambda (-1)^{d-k} b_{d-k} \end{aligned}$$

The above, in particular, asserts that

$$1 = |\lambda|^2 (-1)^{d+1}$$

and we must have d odd, $|\lambda| = 1$ and $b_k = \overline{\lambda} (-1)^k \overline{a_{d-k}}$. □

Let us consider the sets

$$A_d = \{\phi \in \text{Rat}_d : \tau \in \text{Aut}(\phi)\}$$

$$\mathcal{A}_d = \{[\phi] \in M_d : \phi \in A_d\}$$

$$A_d^{\mathbb{R}} = \{\phi \in A_d : \phi \text{ is real}\}$$

$$\mathcal{A}_d^{\mathbb{R}} = \{[\phi] \in M_d : \phi \in A_d^{\mathbb{R}}\}$$

In this way, $\mathcal{A}_d - \mathcal{A}_d^{\mathbb{R}}$ is the locus in M_d consisting of classes of equivalences of pseudo-real rational maps of degree d admitting an imaginary reflection as an antiholomorphic automorphism.

Let $V_d \subset \mathbb{P}_{\mathbb{C}}^d$ be the hypersurface defined by the zero locus of the resultant of the two polynomials $P(z) = \sum_{k=0}^d a_k z^k$ and $Q(z) = \sum_{k=0}^d (-1)^k \overline{a_{d-k}} z^k$, where the points in $\mathbb{P}_{\mathbb{C}}^d$ is $[a_0 : a_1 : \dots : a_d]$.

A direct consequence of the above is the following.

Theorem 8. *If $d \geq 1$ is odd, then we may identify A_d with $S^1 \times (\mathbb{P}_{\mathbb{C}}^d - V_d)$; the identification given by*

$$\begin{aligned} S^1 \times (\mathbb{P}_{\mathbb{C}}^d - V_d) &\rightarrow A_d \\ (e^{i\theta}, [a_0 : \dots : a_d]) &\mapsto \phi(z) = \frac{\sum_{k=0}^d a_k z^k}{\sum_{k=0}^d (-1)^k e^{i\theta} \overline{a_{d-k}} z^k} \end{aligned}$$

In particular, A_d and \mathcal{A}_d are connected,

$$\dim_{\mathbb{R}}(A_d) = 2d + 1$$

$$\dim_{\mathbb{R}}(\mathcal{A}_d) = 2d - 2.$$

Remark 4. *In order to see the real dimension of \mathcal{A}_d , one only needs to observe that the normalizer, in $\text{PSL}_2(\mathbb{C})$, of τ is given by the subgroup of Möbius transformations*

$$A(z) = \frac{az + b}{-\overline{b}z + \overline{a}}$$

where $|a|^2 + |b|^2 = 1$.

Let us now identify those rational maps in A_d belonging to $A_d^{\mathbb{R}}$, that is, the real rational maps inside A_d . As for $d = 1$ there are no pseudo-real maps, we will assume from now on that $d \geq 3$ odd.

If we have $\phi \in A_d^{\mathbb{R}}$, then there is a reflection ρ as antiholomorphic automorphism of ϕ . Let us denote by Σ_ρ the circle of fixed points of ρ .

Lemma 1. *Either $\Sigma_\rho = S^1$ or it is an Euclidean line through 0 union ∞ .*

Proof. If Σ_ρ is not of the desired form, then the Möbius transformation $\rho \circ \tau$ is a loxodromic transformation. This is a contradiction to the fact that rational maps of degree at least two have finite group of automorphisms. \square

Case: $\Sigma_\rho = S^1$. In this case, $\rho(z) = 1/\bar{z}$ and $T(z) = -z$ is holomorphic automorphism of ϕ . So, $\phi(z) = z\psi(z^2)$, for a suitable rational map $\psi \in \text{Rat}_r$, where $r \in \{(d \pm 1)/2\}$. In this case, the map ψ should satisfy the equality

$$\psi(z) = \frac{1}{\overline{\psi(1/z)}}$$

Equivalently, if we write

$$\phi(z) = \frac{\sum_{k=0}^d a_k z^k}{\sum_{k=0}^d (-1)^k e^{i\theta} \overline{a_{d-k}} z^k},$$

then either (i) $a_k = 0$ for k odd or (ii) $a_k = 0$ for k even. In this way, these rational maps corresponds to the sublocus (of real dimension d) of $S^1 \times (\mathbb{P}_{\mathbb{C}}^d - V_d)$ (as in Theorem 8) given as

$$S^1 \times \{[0 : a_1 : 0 : a_3 : \cdots : 0 : a_d] \in \mathbb{P}_{\mathbb{C}}^d - V_d\} \cup S^1 \times \{[a_0 : 0 : a_2 : \cdots : 0 : a_{d-1} : 0] \in \mathbb{P}_{\mathbb{C}}^d - V_d\}.$$

Case: Σ_ρ is an Euclidean line through 0 union ∞ . In this situation, $\rho(z) = e^{2\alpha i} \bar{z}$. Again writing ϕ as above, the equality $\phi \circ \rho = \rho \circ \phi$ ensures that there is some $\mu \neq 0$ so that

$$\begin{aligned} \overline{a_k} &= \mu a_k e^{2(k-1)\alpha i} \\ a_{d-k} &= \mu e^{2(\theta+k\alpha)i} \overline{a_{d-k}} \end{aligned}$$

from where we obtain that $|\mu| = 1$.

If $a_0 \neq 0$, then the above (for $k = 0, d$) asserts that $\mu = e^{((1-d)\alpha - \theta)i}$ and

$$a_k = |a_k| e^{((d+1-2k)\alpha + \theta)i/2}$$

So the only free parameters are $r_k = |a_k|$, $k = 0, \dots, d$. We obtain in this way a sublocus of real dimension $d + 1$ (up to projectivization we may assume $r_0 = 1$) inside A_d .

The situation is similar for $a_d \neq 0$.

Theorem 9. *If $d \geq 3$ is odd, then*

- (1) $\dim_{\mathbb{R}}(A_d^{\mathbb{R}}) = d + 1$.
- (2) $\dim_{\mathbb{R}}(\mathcal{A}_d^{\mathbb{R}}) = d - 2$.
- (3) $A_d - A_d^{\mathbb{R}}$ and $\mathcal{A}_d - \mathcal{A}_d^{\mathbb{R}}$ are connected.

Corollary 3. *If $d \geq 3$ is odd, then the sublocus in M_d of pseudo-real rational maps admitting an imaginary reflection as automorphism is connected and of real dimension $d - 2$.*

Remark 5. *The real projective plane $\mathbb{P}_{\mathbb{R}}^2$ can be identified with the Klein surface $\widehat{\mathbb{C}}/\langle \tau(z) = -1/\bar{z} \rangle$. If $\phi \in A_d$, then it defines a (dianalytic) rational map $\widehat{\phi} : \mathbb{P}_{\mathbb{R}}^2 \rightarrow \mathbb{P}_{\mathbb{R}}^2$. Some of these rational maps (the bicritical ones) were studied in [2].*

5.2. case $n \geq 3$ odd. In the case that n is odd, then $\tau_n^n(z) = \tau(z) = -1/\bar{z}$. This case was already worked above (note that Theorem 3 asserts that in this case ϕ cannot be pseudo-real).

5.3. case $n \geq 2$ even. In this case, $T(z) = \tau_n^2(z) = \omega_n z$. As seen from Theorem 7, $\phi(z) = z\psi(z^n)$ for a suitable rational map $\psi \in \text{Rat}_r$, where $r \in \{(d-1)/n, d/n, (d+1)/n\}$.

The equality $\tau_n \circ \phi = \phi \circ \tau_n$ is equivalent to have

$$\psi(-1/\bar{z}) = 1/\bar{\psi}(z).$$

If we write

$$\psi(z) = \frac{\sum_{k=0}^r a_k z^k}{\sum_{k=0}^r b_k z^k} \in \text{Rat}_r,$$

then the above equality is equivalent to the existence of some $\lambda \neq 0$ so that

$$\bar{b}_k = \lambda(-1)^{r-k} a_{r-k},$$

$$\bar{a}_k = \lambda(-1)^{r-k} b_{r-k}.$$

As the last equality above is equivalent to

$$\bar{b}_k = \bar{\lambda}^{-1}(-1)^k a_{r-k},$$

the above, in particular, ensures that $|\lambda| = 1$, r even and $b_k = \bar{\lambda}(-1)^k \overline{a_{r-k}}$.

As we know that $n \geq 2$ is even, the parity on r ensures one of the followings:

- (1) if $r = (d-1)/n$, then $d \equiv 1 \pmod{4}$;
- (2) if $r = dn$, then $d \equiv 0 \pmod{4}$;
- (3) if $r = (d+1)/n$, then $d \equiv 3 \pmod{4}$.

Summarizing all the above is the following.

Theorem 10. *If a rational map of degree d admits an antiholomorphic automorphism of order $2n$, where $n \geq 2$ is even, then $d \not\equiv 2 \pmod{4}$. Moreover, in the affirmative case, it is conjugated to one of the form $\phi(z) = z\psi(z^n)$, for a suitable rational map $\psi \in \text{Rat}_r$ where $r \in \{(d-1)/n, d/n, (d+1)/n\}$ is even, admitting the antiholomorphic automorphism $\tau(z) = \frac{\omega_{2n}}{\bar{z}}$, where $\omega_{2n} = e^{\pi i/n}$. If*

$$\psi(z) = \frac{\sum_{k=0}^r a_k z^k}{\sum_{k=0}^r b_k z^k} \in \text{Rat}_r,$$

then there is some $\theta \in \mathbb{R}$ so that, for all k , $b_k = (-1)^k e^{i\theta} \overline{a_{r-k}}$.

In particular, if we denote by $B_d(n)$ the subset of Rat_d admitting τ above as automorphism, then we may identify it with $S^1 \times (\mathbb{P}_{\mathbb{C}}^r - W_d)$ (where W_d is certain hypersurface); the identification given by

$$S^1 \times (\mathbb{P}_{\mathbb{C}}^r - V_d) \rightarrow B_d(n)$$

$$(e^{i\theta}, [a_0 : \cdots : a_r]) \mapsto \phi(z) = z \frac{\sum_{k=0}^r a_k z^k}{\sum_{k=0}^r (-1)^k e^{i\theta} \overline{a_{r-k}} z^k}$$

In particular, $B_d(n)$ and $\mathcal{B}_d(n) = \{[\phi] \in \mathcal{M}_d : \phi \in B_d(n)\}$ are connected,

$$\dim_{\mathbb{R}}(A_d) = 2r + 1$$

$$\dim_{\mathbb{R}}(\mathcal{A}_d) = 2r - 2.$$

Corollary 4. *If $d \geq 3$, then the sublocus in \mathcal{M}_d of rational maps admitting an antiholomorphic automorphism of order $2n$, where $n \geq 2$, is connected and of real dimension $2r - 2$.*

5.4. Proof of Theorem 5. We have seen that the locus \mathcal{A}_d is connected. Theorem 10 asserts the existence of a lot of pseudo-real rational maps admitting an antiholomorphic automorphism of order $2n$, for some $n \geq 2$ even. So, in order to prove that $\mathcal{B}_d^{\mathbb{R}}$ is non-connected, we only need to prove that there is no $[\phi] \in \mathcal{A}_d$ so that ϕ admits an antiholomorphic automorphism of order $2n$, for some $n \geq 2$ even.

Let us assume $\phi \in A_d$ admits the antiholomorphic automorphism η of order $2n$, where $n \geq 2$ is even. Then $L = \eta^2$ is holomorphic automorphism of ϕ . As $\tau(z) = -1/\bar{z}$ is antiholomorphic automorphism of ϕ and $\text{Aut}(\phi)$ is cyclic, τ must permute both fixed points of T . We may conjugate by a suitable Möbius transformation that normalizes τ in order to assume that both fixed points of T are ∞ and 0 . In this way, we may assume that $\eta(z) = \omega_{2n}/\bar{z}$. But in this case $T(z) = \omega_n z$ and $T^{n/2}(z) = -z$. In this way, $T^{n/2} \circ \tau(z) = 1/\bar{z}$ is a reflection being an automorphism of ϕ , a contradiction.

5.5. On the number of connected components of $\mathcal{B}_d^{\mathbb{R}}$. We have seen, from Theorem 5, that the locus $\mathcal{B}_d^{\mathbb{R}}$ is non-connected. We may wonder in the number of its connected components.

Let us denote by $\mathcal{B}_d^{\mathbb{R}}(n) \subset \mathcal{B}_d^{\mathbb{R}}$ the sublocus consisting on (classes of) pseudo-real maps admitting an anticonformal automorphism of order $2n$.

Clearly,

$$\bigcup_{n \geq 1 \text{ odd}} \mathcal{B}_d^{\mathbb{R}}(n) \subset \mathcal{A}_d$$

and, from the proof of Theorem 5 that in fact

$$\bigcup_{n \geq 1 \text{ odd}} \mathcal{B}_d^{\mathbb{R}}(n) = \mathcal{A}_d.$$

If n is even, then we may write $n = 2^s q$, where $q \geq 1$ is odd. Then clearly, $\mathcal{B}_d^{\mathbb{R}}(n) \subset \mathcal{B}_d^{\mathbb{R}}(2^s)$.

Now, if $[\phi] \in \mathcal{B}_d^{\mathbb{R}}(2^s) \cap \mathcal{B}_d^{\mathbb{R}}(2^t)$, where $s \neq t$, then (up to conjugation by a suitable Möbius transformation) we may assume that ϕ has the following antiholomorphic automorphisms

$$\tau_1(z) = \frac{\omega_{2^{s+1}}}{\bar{z}}, \quad \tau_2(z) = \frac{\omega_{2^{t+1}}}{\bar{z}}.$$

Assume that $s < t$. Take $\alpha = 2^{t-1} - 2^{t-s-1}$. Then $\tau_2^{2\alpha} \circ \tau_1(z) = -1/\bar{z}$. This will get a contradiction to the fact that \mathcal{A}_d is disjoint from $\mathcal{B}_d^{\mathbb{R}}(2^s)$ (as seen in the proof of Theorem 5).

As a consequence, the number of connected components of $\mathcal{B}_d^{\mathbb{R}}$ is equal to the number of integers $s \geq 0$ so that there is a rational map of degree d admitting an antiholomorphic automorphism of order 2^{s+1} .

6. PROOF OF THEOREMS 2 AND THEOREM 3

6.1. Proof of Theorem 2. If ϕ is pseudo-real, then necessarily there is some $Q \in \widehat{\text{Aut}}(\phi)$ being antiholomorphic. Since $\text{Aut}(\phi)$ is trivial, then $Q^2 = I$. Since ϕ is not real, Q cannot be a reflection; so it is an imaginary reflection. In the other direction, if there is some imaginary reflection $Q \in \widehat{\text{Aut}}(\phi)$, then there is no reflection $R \in \widehat{\text{Aut}}(\phi)$ as in such a case $R \circ Q \in \text{Aut}(\phi)$ is different from the identity.

6.2. Proof of Theorem 3. Let us now consider a rational map $\phi \in \text{Rat}_d$, $d \geq 3$ odd, with $\text{Aut}(\phi) \cong \mathbb{Z}_n$, $n \geq 2$. Up to equivalence, we may assume that $\phi(z) = z\psi(z^n)$, where ψ is some rational map of degree $r \geq 1$, $\text{Aut}(\phi) = \langle T(z) = \omega_n z \rangle \cong \mathbb{Z}_n$, where $\omega_n = e^{2\pi i/n}$.

Let us recall that the map ϕ is pseudo-real if and only if it admits an antiholomorphic automorphism and it has no reflections as automorphisms.

Let Q be an antiholomorphic automorphism of ϕ . As $Q^2 \in \text{Aut}(\phi)$, we must have that Q keeps invariant the set $\{\infty, 0\}$. In this way, we must have one of two possibilities.

- (1) $Q(0) = 0$ and $Q(\infty) = \infty$; so $Q(z) = \beta\bar{z}$, where $|\beta| = 1$, is a reflection. In this case, ϕ is real, and

$$\psi(z) = \bar{\psi}(z/\beta^n).$$

In order to rule out this possibility, the rational map ψ must satisfy that

$$\psi(z) \neq \bar{\psi}(e^{i\theta}z), \quad \forall \theta \in \mathbb{R}.$$

- (2) $Q(0) = \infty$ and $Q(\infty) = 0$; so $Q(z) = \alpha/\bar{z}$, where $\alpha/\bar{\alpha} = \omega_n^k$ for some $k = 0, 1, \dots, n-1$. This is equivalent for ψ to satisfy the equality

$$\psi(z)\bar{\psi}(\alpha^n/z) = 1.$$

We may conjugate ϕ by $L(z) = \lambda z$, where $|\lambda|^2 = 1/|\alpha|$ in order to assume that $\alpha = e^{i\varphi}$.

All elements of $\widehat{\text{Aut}}(\phi) - \text{Aut}(\phi)$ are of the form

$$L(z) = \frac{\alpha\omega_n^s}{\bar{z}},$$

where $s \in \{0, 1, \dots, n-1\}$. So, if $\varphi \neq 2s\pi/n$, for every $s = 0, 1, \dots, n-1$, then ϕ will be pseudo-real.

7. AN EXAMPLE

Let us see that there is no pseudo-real rational map of degree $d = 3$ admitting a non-trivial antiholomorphic automorphism.

Let us assume there is some $\phi \in \text{Rat}_3$ pseudo-real admitting a holomorphic automorphism T of order $n \geq 2$. Up to conjugation, we may assume that $T(z) = \omega_n z$ and $\phi(z) = z\psi(z^n)$, where $r = \deg(\psi)$ and $d \in \{nr-1, nr, nr+1\}$. This asserts that $(n, r) \in \{(2, 1), (2, 2), (3, 1), (4, 1)\}$.

7.0.1. *The case $n = 2$.* An antiholomorphic automorphism Q of ϕ must keep invariant the set $\{0, \infty\}$. As ϕ is pseudo-real, $Q(0) \neq 0$; otherwise Q must be a reflection. It follows that $Q(z) = \alpha/\bar{z}$, where $\alpha \notin [0, +\infty)$ satisfies that $\alpha/\bar{\alpha} \in \{-1, 1\}$; in particular either (i) $\alpha < 0$ or (ii) $\alpha = is$, for some $s \in \mathbb{R} - \{0\}$. Since $(T \circ Q)(z) = -\alpha/\bar{z}$ is also an antiholomorphic automorphism of ϕ , the case (i) is neither possible; otherwise ϕ will have a reflection. In this way, $Q(z) = is/\bar{z}$, for a suitable $s \in \mathbb{R} - \{0\}$.

(1) If $r = 1$, that is, $\psi(z) = (a + bz)/(c + dz)$, where $ad - bc \neq 0, bc \neq 0$,

$$\phi(z) = \frac{z(a + bz^2)}{c + dz^2}.$$

The equality $Q \circ \phi \circ Q^{-1} = \phi$ ensures that $a = b = c = d = 0$, a contradiction.

(2) Now, let us consider the case $r = 2$, that is

$$\phi(z) = \frac{a + bz^2}{cz + dz^3}, \quad ad \neq 0.$$

The equality $Q \circ \phi \circ Q^{-1} = \phi$ again ensures that $a = b = c = d = 0$, a contradiction.

7.0.2. *The case $n = 3$.* In this case,

$$\phi(z) = \frac{az}{b + cz^3}, \quad abc \neq 0.$$

If we assume there is an antiholomorphic automorphism of ϕ , different from a reflection, then $Q(z) = \alpha/\bar{z}$, where $\alpha/\bar{\alpha} \in \{1, \omega_3, \omega_3^2\}$. But the condition $\phi = Q \circ \phi \circ Q^{-1}$ will be equivalent to have the equality

$$\frac{az}{b + cz^3} = \frac{\bar{b}z^3 + \bar{c}\alpha^3}{\bar{a}z^2}$$

which is not possible.

7.0.3. *The case $n = 4$.* In this case $\psi(z) = a/z$ and $\phi(z) = a/z^3$. Up to conjugation by a dilation, we may also assume $a = 1$; but in this case ϕ is seen to be real.

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